2-restricted Lie algebras associated with right-angled Coxeter groups

Temurbek Rahmatullaev

First, I will try to establish the relationship between the problem of describing the associated Lie algebra for right-angled Coxeter groups and toric topology. The primary focus of the work was to provide an explicit description of the 2-restricted version of the associated Lie algebra for these groups.

To achieve this, we will introduce the 2-restricted analogue of lower central series, briefly describe the properties of the associated algebra - the 2-restricted Lie algebra. A key result we will use is Quillen's theorem. This theorem connects the universal enveloping algebra of a 2-restricted Lie algebra with the graded ring of the group ring.

The theory we develop will be applied to demonstrate the isomorphism between the 2-restricted associated Lie algebra of a Coxeter group and the 2-graph Lie algebra:

$$L_{\mathcal{K}}^{[2]} = FL_{\mathbb{Z}_2}^{[2]}(\mathcal{K}^0)/\langle [v_i, v_j] = 0, \{i, j\} \in \mathcal{K}; \quad v_i^{[2]} = 0, \forall i \in \mathcal{K}^0 \rangle.$$

As a consequence of this isomorphism, for flag complexes $\mathcal{K}$, we get a connection between the fundamental group of the polyhedral power of a real infinite-dimensional projective space and the Pontryagin algebra of the polyhedral power of a complex infinite-dimensional projective space:

$$\overline{U}(\mathrm{gr}^{[2]}\,\pi_1((\mathbb{R}P^\infty)^{\mathcal{K}})) = H_*(\Omega(\mathbb{C}P^\infty)^{\mathcal{K}}; \mathbb{Z}_2).$$

Objects of study

Let $\mathcal{K}$ be a simplicial complex on vertex set $[m] = \{1, \ldots, m\}$. For any sequence of CW-pairs $(\underline{X}, \underline{A}) = ((X_1, A_1), \ldots, (X_m, A_m))$, consider the polyhedral product:

$$(\boldsymbol{X}, \boldsymbol{A})^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} (\boldsymbol{X}, \boldsymbol{A})^I = \bigcup_{I \in \mathcal{K}} \left(\prod_{i \in I} X_i \times \prod_{i \notin I} A_i \right).$$

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The most important examples for us are:

- $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ – *moment-angle complexes*
- $\mathcal{R}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ – *real moment-angle complexes*
- $\mathcal{L}_{\mathcal{K}} = (\mathbb{R}, \mathbb{Z})^{\mathcal{K}} = \bigcup_{I \in \mathcal{K}} (\mathbb{R}, \mathbb{Z})^I$

Objects of study

Let G be a group. *Central series on* G is a sequence of subgroups $\mathcal{G} = \{\mathcal{G}_k\}_{k \geqslant 1}$ such that:

1. $\mathcal{G}_1 = G$
2. $\mathcal{G}_{k+1} < \mathcal{G}_k$
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Theorem

The bracket defined as follows

$$[\sum_i x_i \mathcal{G}_{i+1}, \sum_j y_j \mathcal{G}_{j+1}] = \sum_{i,j} (x_i, y_j) \mathcal{G}_{i+j+1}$$

defines the structure of a graded Lie ring on $\operatorname{gr} \mathcal{G} = \bigoplus \mathcal{G}_i / \mathcal{G}_{i+1}$.

Lower central series is defined recursively: $\gamma_n(G) = (\gamma_{n-1}(G), G)$.

Parallel of the real and complex case

For real and complex cases of moment-angle complexes two parallel (homology and homotopy) theories rise. One of the question on the way of understanding the connection is calculating the Lie algebra associated to the Coxeter groups.

Here and below assume $\mathcal{K}$ is flag.

Parallel of the real and complex case

Case of $\mathcal{Z}_\mathcal{K}$ [Grb+15]

Proposition

There is a homotopy fibration:

$$\mathcal{Z}_\mathcal{K} \to (\mathbb{C}P^\infty)^\mathcal{K} \to (\mathbb{C}P^\infty)^m.$$

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Considering loop homology, if k is field or $\mathbb{Z}$, we obtain a split exact sequence of (noncommutative) algebras:

$$1 \to H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathrm{k}) \to H_*(\Omega(\mathbb{C}P^{\infty})^{\mathcal{K}}; \mathrm{k}) \to \Lambda[m] \to 1.$$

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If k is a field, then there is an explicit description:

$$H_*(\Omega(\mathbb{C}P^{\infty})^{\mathcal{K}}; \mathrm{k}) = \mathrm{Ext}_{\mathrm{k}[\mathcal{K}]}(\mathrm{k}, \mathrm{k}) \cong$$

$$\cong \frac{T\langle u_1, \dots, u_m \rangle}{(u_i^2 = 0, \forall i; u_i u_j + u_j u_i = 0, \text{ for } \{i, j\} \in \mathcal{K})}$$

Proposition

There is a homotopy fibration:

$$\mathcal{R}_{\mathcal{K}} \to (\mathbb{R}P^{\infty})^{\mathcal{K}} \to (\mathbb{R}P^{\infty})^{m},$$

moreover, for a flag $\mathcal{K}$ all three spaces are aspherical.

Parallel of the real and complex case

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All topological information is contained in the fundamental groups of spaces. Passing to fundamental groups, we obtain the exact sequence:

$$1 \to \mathrm{RC}_\mathcal{K}' \to \mathrm{RC}_\mathcal{K} \to \mathbb{Z}_2^{\oplus m} \to 1,$$

where $\mathrm{RC}_\mathcal{K} = F(\mathcal{K}^0)/\langle v_i^2 = 1 \text{ for } i \in [m]; v_i v_j = v_j v_i \text{ for } i, j \in \mathcal{K}\rangle$.

General question

It is natural to ask the following problem: is it possible to construct a graded algebra from the group $RC_{\mathcal{K}}$, that would contain homotopy information about $\mathcal{Z}_{\mathcal{K}}$.

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Here comes the second motivating parallel:

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Considering fundamental groups, we obtain:

$$1 \to \mathrm{RA}_{\mathcal{K}}' \to \mathrm{RA}_{\mathcal{K}} \to \mathbb{Z}^{\oplus m} \to 1,$$

where $\mathrm{RA}_{\mathcal{K}} = F(\mathcal{K}^0)/\langle v_i v_j = v_j v_i \Leftrightarrow i, j \in \mathcal{K}\rangle$.

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where $\mathrm{RA}_\mathcal{K} = F(\mathcal{K}^0)/\langle v_i v_j = v_j v_i \Leftrightarrow i, j \in \mathcal{K}\rangle$.
It is known that the Lie ring associated with the LCS of $\mathrm{RA}_\mathcal{K}$ has an explicit description [DK92; Wad16] and is isomorphic to the *Lie graph-ring*:

$$\mathrm{gr}(\gamma(\mathrm{RA}_\mathcal{K})) \cong L_\mathcal{K} = \frac{\mathrm{FL}(\mathcal{K}^0)}{([v_i, v_j] = 0, for\{i, j\} \in \mathcal{K})}.$$

Obstacle

We will call Lie graph-algebra over $\mathbb{Z}_2$ the following:

$$L_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_2 = FL_{\mathbb{Z}_2}(\mathcal{K}^0)/([v_i, v_j] = 0 \Leftrightarrow i, j \in \mathcal{K}).$$

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Proposition

The following natural map is epimorphic, but not monomorphic:

$$e_{\mathrm{RC}_{\mathcal{K}}} : L_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to \operatorname{gr} \gamma(\mathrm{RC}_{\mathcal{K}})$$

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If monomial element $a \in U(L_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Z}_2)$ has degree n, then a^2 (as a monomial element corresponding to a^2 from group) has degree $2n$. But for nonzero monomial element $a \in U(\operatorname{gr}\gamma(\mathrm{RC}_{\mathcal{K}}))$ with degree n, element corresponding to a^2 from group can has degree $n+1$.

p-restricted central series

Definition

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The construction of the minimal inclusion-wise N_p-series for the LCS was introduced by H. Zassenhaus [Zas39].

Definition

For any central series $\{K_i\}_{i\geqslant 1}$ define the *p-restricted central series constructed from* $\{K_i\}_{i\geqslant 1}$ in the following way:

$$K_n^{[p]} = \prod_{mp^j\geqslant n, m\geqslant 1, j\geqslant 0} (K_m)^{p^j}$$

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Notice that for $g \in K_n^{[p]}$, the element $g^p \in K_{np}^{[p]}$. Hence, the operation induced in the p-restricted Lie algebra $\mathrm{gr}(\mathrm{K}^{[\mathrm{p}]})$ is both well-defined and respects the grading.

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For $\overline{g} = gK_{n+1}^{[p]} \in K_n^{[p]}/K_{n+1}^{[p]}$, we can define

$$\overline{g}^{[p]} = \overline{g^p} = g^p K_{np+1}^{[p]} \in K_{np}^{[p]}/K_{np+1}^{[p]}.$$

We will denote $\mathrm{gr}^{[p]}(G) = \mathrm{gr}(\gamma^{[p]}(G))$ as the Lie algebra associated with the p-restricted $\gamma^{[p]}$.

p-restricted Lie algebra

Definition

***p*-restricted Lie algebra** is defined as a Lie algebra L over a field k of characteristic p with the introduced p-operation $x \mapsto x^{[p]}$, such that for all $x, y \in L$:

1. $[x, y^{[p]}] = [x, y, \ldots, y]$
2. $(tx)^{[p]} = t^p x^{[p]}$, $t \in k$
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} i^{-1} s_i(x, y)$, where $s_i(x, y)$ are formal coefficients in front of t^{i-1} in the expression $\mathrm{ad}_x(tx+y)^{p-1} = [x, tx+y, tx+y, ..., tx+y]$ in the associative Lie algebra.

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Theorem ([Laz54])

The Lie algebra associated with the filtration $\{K_n^{[p]}\}$ is a p-restricted Lie algebra.

On the associated graded ring of a group ring[Qui68]

Let R be a field with $\mathrm{char} R = p > 0$, RG a group ring with augmentation homomorphism $\varepsilon : RG \to R$, given as $\varepsilon(\sum r_i g_i) = \sum r_i$, where $r_i \in R$, $g_i \in G$ and $\overline{RG} = \ker \varepsilon$ the corresponding augmentation ideal.

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The algebra associated with filtration by the augmentation ideal will be denoted as follows:

$$\mathrm{gr}(RG) = \bigoplus_{n \geqslant 0} (\overline{RG})^n / (\overline{RG})^{n+1}$$

On the associated graded ring of a group ring[Qui68]

Proposition ([Qui68, Lemma 2.1])

Let $w : L_1 \to L_2$ be a homomorphism of p-Lie algebras over K. Then, w is surjective (injective) if and only if $\overline{U}w : \overline{U}L_1 \to \overline{U}L_2$ is surjective (injective).

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Theorem ([Qui68, Th. 1], [Pas06, Th. VIII.5.2])

There is an isomorphism of graded algebras over R:

$$\overline{U}(\mathrm{gr}^{[p]}(G) \otimes_{\mathbb{Z}} R) \to \mathrm{gr}(RG).$$

Definitions

Definition

If X is a non-empty set, then the free p-bounded Lie algebra $FL^{[p]}(X)$ is defined as a p-Lie algebra generated by the set X, such that any mapping $\phi : X \to G$, where G is a p-Lie algebra, extends to a p-homomorphism $\hat{\phi} : FL^{[p]}(X) \to G$.

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Definition

Let $\mathcal{K}$ be a graph on the set of vertices $\mathcal{K}^0$. Then the p-graph algebra of Lie algebras is defined as:

$$L_{\mathcal{K}}^{[p]} = FL_{\mathbb{Z}_p}^{[p]}(\mathcal{K}^0)/\langle [v_i, v_j] = 0, \{i,j\} \in \mathcal{K}; \quad v_i^{[p]} = 0\rangle$$

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Proposition

The identity mapping id $: \mathcal{K}^0 \to \mathcal{K}^0$ *extends to an epimorphism of 2-Lie algebras* $e_{\mathrm{RC}_{\mathcal{K}}}^{[2]} : L_{\mathcal{K}}^{[2]} \to \mathrm{gr}^{[2]}(\mathrm{RC}_{\mathcal{K}})$.

The main result

Theorem

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Recalling the motivation behind the study of the lower central series, the fact that $\mathrm{RC}_{\mathcal{K}} = \pi_1((\mathbb{R}P^{\infty})^{\mathcal{K}})$, and that for k – a field, we have

$$H_*(\Omega(\mathbb{C}P)^{\mathcal{K}}; \mathrm{k}) = \operatorname{Ext}_{\mathrm{k}[\mathcal{K}]}(\mathrm{k}, \mathrm{k}) \cong \frac{T\langle u_1, \ldots, u_m \rangle}{(u_i^2 = 0, u_i u_j + u_j u_i = 0, \Leftrightarrow \{i, j\} \in \mathcal{K})}$$

at the level of universal enveloping algebras, we can formulate the following

Corollary

There exists an isomorphism of associative algebras

$$\overline{U}(\operatorname{gr}^{[2]} \pi_1((\mathbb{R}P^{\infty})^{\mathcal{K}})) = H_*(\Omega(\mathbb{C}P^{\infty})^{\mathcal{K}}; \mathbb{Z}_2)$$

Steps of proof

Proposition

$$\mathbb{Z}_2\mathrm{RC}_{\mathcal{K}} \simeq \mathbb{T}_{\mathbb{Z}_2}(\mathcal{K}^0)/(v_i^2 = 1, \forall i; \quad v_i v_j v_i v_j = 1 \Leftrightarrow \{i,j\} \in \mathcal{K})$$

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Theorem

Let the graph $\mathcal{K}$ be defined on the set of vertices $[m]$, and let the generators of $L_{\mathcal{K}}^{[2]}$ be $\{v_i\}_{i=0}^m$. Then,

$$\overline{U}(L_{\mathcal{K}}^{[2]}) = \mathbb{T}_{\mathbb{Z}_2}(a_0, \dots, a_m)/(a_i^2 = 0, \forall i; \quad a_i a_j + a_j a_i = 0 \Leftrightarrow \{i,j\} \in \mathcal{K})$$

Steps of proof

Consider two augmented algebras

$$\mathbb{Z}_2 RC_{\mathcal{K}} \cong \mathbb{T}_{\mathbb{Z}_2}(v_1, \ldots, v_m)/(v_i^2 - 1 = 0, \forall i; \quad v_i v_j v_i v_j - 1 = 0 \Leftrightarrow \{i, j\} \in \mathcal{K})$$

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Augmentations are defined on them as follows:

1. $\varepsilon : \mathbb{Z}_2 RC_{\mathcal{K}} \to \mathbb{Z}_2$ is defined as $\varepsilon : v_i \mapsto 1$.
2. $\tilde{\varepsilon} : \overline{U}(L_{\mathcal{K}}^{[2]}) \to \mathbb{Z}_2$ is defined as $\tilde{\varepsilon} : a_i \mapsto 0$.

Theorem

The mapping $\tilde{\mu} : v_i \mapsto a_i + 1$ establishes isomorphisms of augmented algebras $\mathbb{Z}_2 RC_{\mathcal{K}} \simeq \overline{U}(L_{\mathcal{K}}^{[2]})$.

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The mapping $\tilde{\mu} : v_i \mapsto a_i + 1$ *establishes isomorphisms of augmented algebras* $\mathbb{Z}_2\mathrm{RC}_{\mathcal{K}} \simeq \overline{U}(L_{\mathcal{K}}^{[2]})$.

Note that the established isomorphism between $\mathbb{Z}_2\mathrm{RC}_{\mathcal{K}}$ and the connected graded algebra $\overline{U}(L_{\mathcal{K}}^{[2]})$ gives a grading on $\mathbb{Z}_2\mathrm{RC}_{\mathcal{K}}$. Filtration we get equals to the filtration given by degrees of augmentation ideal as it is so in $\overline{U}(L_{\mathcal{K}}^{[2]})$.

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Corollary

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Corollary

$$\overline{U}(\mathrm{gr}^{[p]}(\mathrm{RC}_{\mathcal{K}}) \otimes_{\mathbb{Z}} \mathbb{Z}_2) \cong \mathrm{gr}(\mathbb{Z}_2\mathrm{RC}_{\mathcal{K}}) \cong \mathbb{Z}_2\mathrm{RC}_{\mathcal{K}} \cong \overline{U}(L_{\mathcal{K}}^{[2]})$$

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